CS-466/566: Math for AI

Module 04: Logistic Regression

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Classification vs. Regression

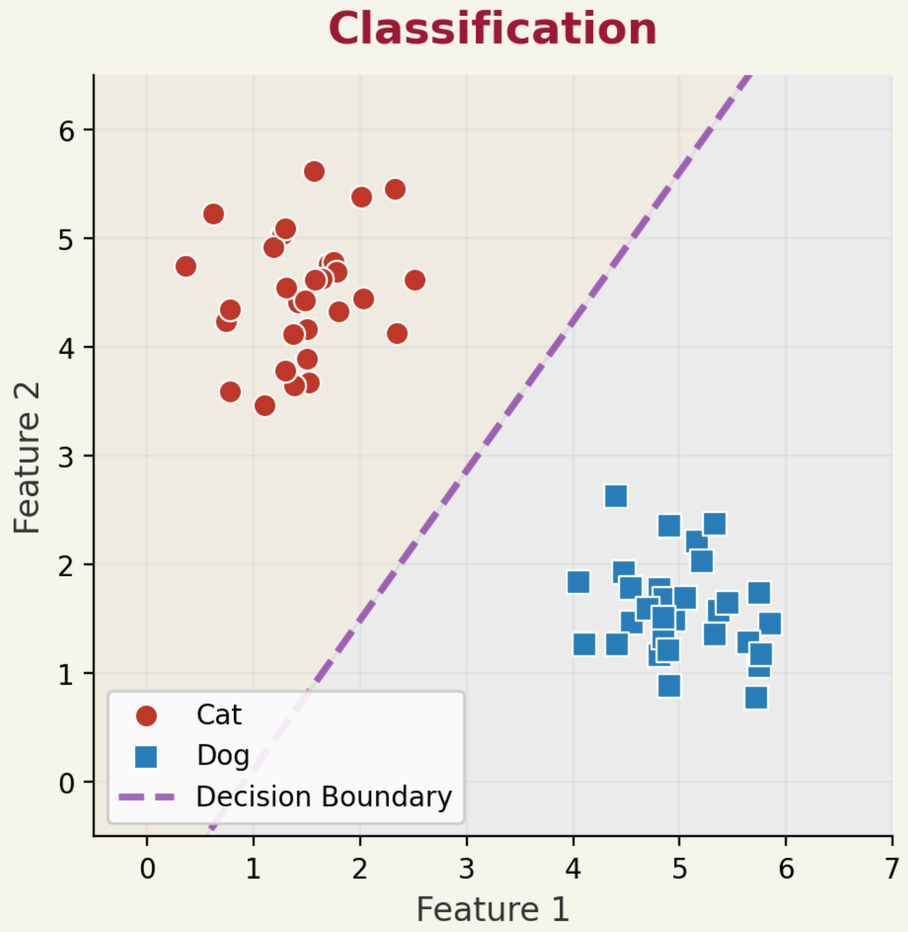
🏷️ Classification Models

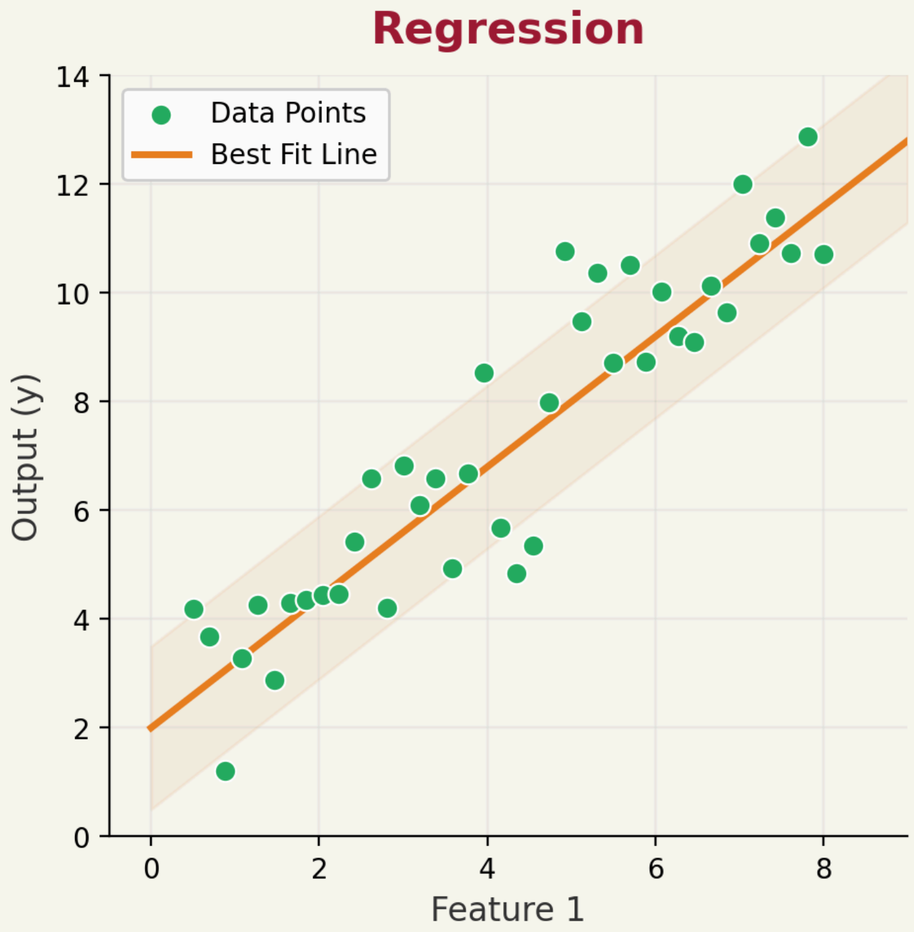
- Predict a **categorical** output
- The output is **discrete** (class labels)
- 🐱🐶 Type of animal (cat or dog)
- 📧 Email spam detection

📈 Regression Models

- Predict a **numerical** output
- The output is a **continuous number**
- 🏠 Housing prices
- ⚖️ Predicting the weight of an object

Classification vs. Regression



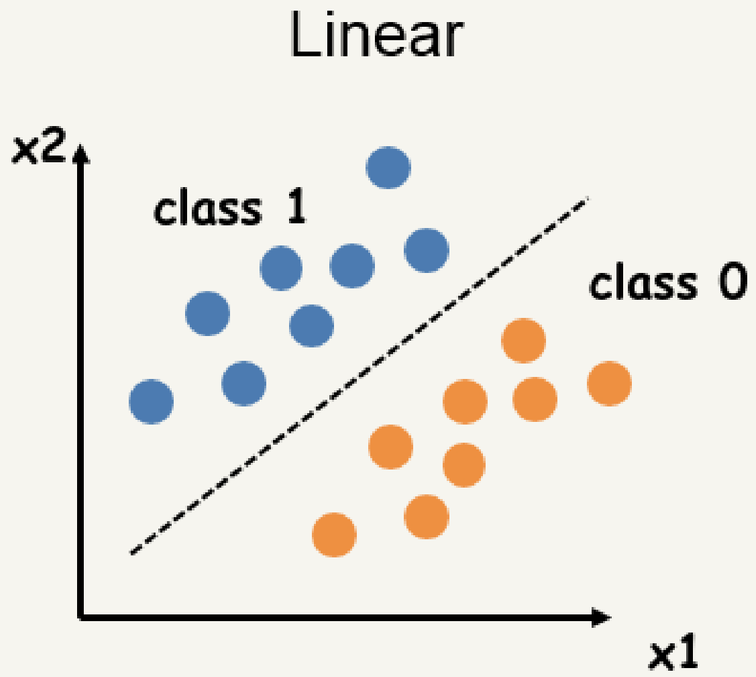


Will You Pass the Exam?

📋 The Scenario

- A professor is tracking student outcomes
- Measures: **study hours** and **sleep hours**
- Goal: predict **Pass** ✅ or **Fail** ❌

Study (x_1)	Sleep (x_2)	Result
5	7	✅ Pass
3	8	✅ Pass
1	3	❌ Fail
2	2	❌ Fail



Key Question: How do we draw a line to separate the classes?

The Decision Boundary

We want to find a **line** (decision boundary) that separates the two classes.

Decision Boundary:

$$w_1 x_1 + w_2 x_2 + w_0 = 0$$

- **Pass** (Class 1): $w_1 x_1 + w_2 x_2 + w_0 > 0$
- **Fail** (Class 0): $w_1 x_1 + w_2 x_2 + w_0 < 0$

In general (more than 2 features):

$$w_1 x_1 + w_2 x_2 + \cdots + w_n x_n + w_0 = 0$$

The model is still **linear**, but the boundary is a **hyperplane** in n dimensions.

Another Dataset Example

In practice, data is rarely perfectly separable.

Goal: Find the **most general** linear boundary that minimizes classification error.

- Points near the boundary may be misclassified
- We want the boundary that makes the **fewest mistakes** overall

Key insight: We need a model that outputs a **confidence score** (probability), not just a hard 0/1 decision.

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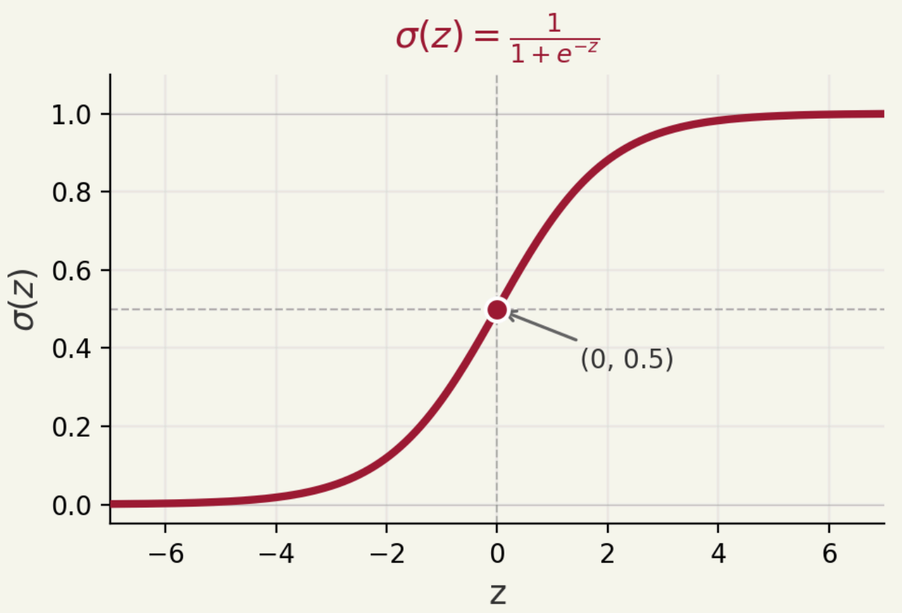
From Scores to Probabilities

Let $z = w_1 x_1 + w_2 x_2 + \cdots + w_n x_n + w_0$ be the linear score.

We need a function that maps $z \in (-\infty, +\infty)$ to a probability in $[0, 1]$.

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

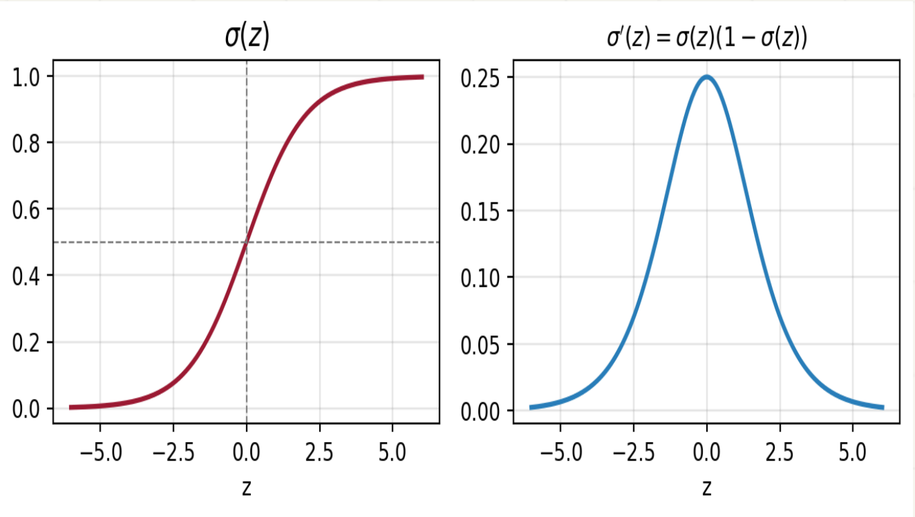
Input z	$\sigma(z)$	Interpretation
$-\infty$	0	Definitely Class 0
0	0.5	Uncertain
$+\infty$	1	Definitely Class 1



Sigmoid: Shape and Derivative

Properties of $\sigma(z)$:

- Output is always in $[0, 1]$ ✅
- Smooth and differentiable everywhere ✅
- Symmetric around $(0, 0.5)$
- Maps **any** real number to a probability



Simple and elegant derivative:

$$\frac{d\sigma}{dz} = \sigma(z)\big(1 - \sigma(z)\big)$$

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The Logistic Regression Model

Logistic Regression is a **linear classifier** that outputs a **probability**.

$$\hat{y} = \sigma(z) = \sigma\left(\sum_{i=0}^{n} w_i x_i\right) = \frac{1}{1 + e^{-\mathbf{w}^T \mathbf{x}}}$$

Prediction rule:

- If $\hat{y} > 0.5$ → predict **Class 1** (Pass ✅)
- If $\hat{y} \leq 0.5$ → predict **Class 0** (Fail ❌)

This is equivalent to: if $z > 0$ → Class 1, if $z \leq 0$ → Class 0.

Why "Regression"? Because the internal score z is computed via **linear regression** — logistic regression wraps it with a sigmoid to produce probabilities.

Example: Predicting Exam Results

Suppose the model has learned: $\hat{y} = \sigma\left(x_{\text{study}} + 2\,x_{\text{sleep}} - 8\right)$

Student	Study (x_1)	Sleep (x_2)	z	$\hat{y} = \sigma(z)$	Prediction
1	5	7	$5 + 14 - 8 = 11$	≈ 1.00	✅ Pass
2	3	8	$3 + 16 - 8 = 11$	≈ 1.00	✅ Pass
3	1	3	$1 + 6 - 8 = -1$	0.269	❌ Fail
4	2	2	$2 + 4 - 8 = -2$	0.119	❌ Fail

The model outputs a **confidence**: Student 4 has only a 11.9% chance of passing — much more informative than a hard 0/1 label!

Logistic Regression Summary

Component	Formula / Description
Linear Score	$z = \mathbf{w}^T \mathbf{x} = w_1 x_1 + \cdots + w_n x_n + w_0$
Output (probability)	$\hat{y} = \sigma(z) = \frac{1}{1+e^{-z}} \in [0, 1]$
Decision Boundary	$z = 0 \iff \hat{y} = 0.5$
Class 1	$\hat{y} > 0.5$ (i.e., $z > 0$)
Class 0	$\hat{y} \leq 0.5$ (i.e., $z \leq 0$)

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Why Not Use Squared Loss?

Since $\hat{y} \in [0, 1]$ is a **probability**, squared loss $(y - \hat{y})^2$ is technically valid but leads to a **non-convex** optimization landscape with many local minima.

We need a loss designed for **probability outputs** → **Log Loss** (Cross-Entropy Loss)

Intuition:

- If $y = 1$ and $\hat{y} \approx 1$ → loss should be **small** ✅
- If $y = 1$ and $\hat{y} \approx 0$ → loss should be **large** ❌
- We want a function that **heavily penalizes confident wrong predictions**

Log Loss (Cross-Entropy Loss)

When $y = 1$:

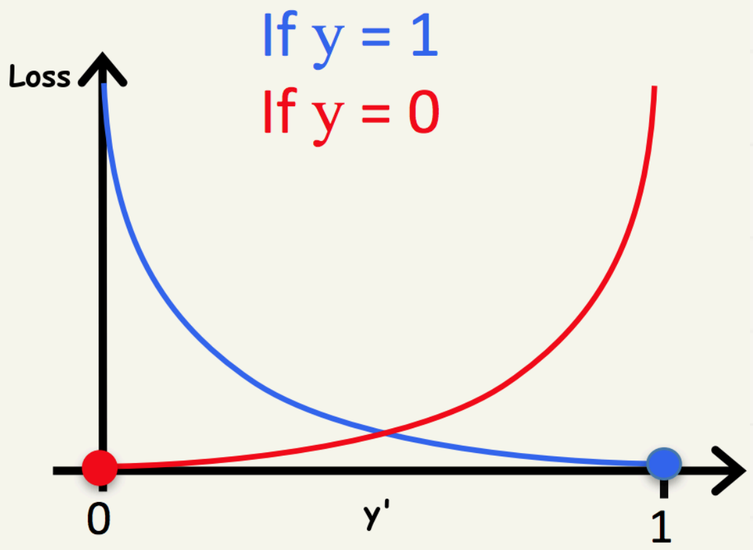
$$\ell = -\log(\hat{y})$$

- $\hat{y} \to 1$: loss $\to 0$ ✅
- $\hat{y} \to 0$: loss $\to \infty$ ❌

When $y = 0$:

$$\ell = -\log(1 - \hat{y})$$

- $\hat{y} \to 0$: loss $\to 0$ ✅
- $\hat{y} \to 1$: loss $\to \infty$ ❌



Unified: $$\ell(y, \hat{y}) = -y \log(\hat{y}) - (1 - y) \log(1 - \hat{y})$$

Unified Loss Formula

$$\ell(y, \hat{y}) = -y \log(\hat{y}) - (1 - y) \log(1 - \hat{y})$$

Verify the formula:

- When $y = 1$: $\ell = -\log(\hat{y}) - 0 = -\log(\hat{y})$ ✅
- When $y = 0$: $\ell = 0 - \log(1 - \hat{y}) = -\log(1 - \hat{y})$ ✅

Over the full dataset (N examples):

$$\square = -\frac{1}{N} \sum_{i=1}^{N} \left[y^{(i)} \log(\hat{y}^{(i)}) + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)}) \right]$$

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Applying the Chain Rule

We have the chain: $w_i \xrightarrow{z=\mathbf{w}^T\mathbf{x}} z \xrightarrow{\sigma} \hat{y} \xrightarrow{\ell} \text{loss}$

$$\frac{d\ell}{dw_i} = \frac{d\ell}{d\hat{y}} \cdot \frac{d\hat{y}}{dz} \cdot \frac{dz}{dw_i}$$

Step 1 — $\frac{dz}{dw_i}$**:** $\quad \frac{dz}{dw_i} = x_i$

Step 2 — $\frac{d\hat{y}}{dz}$**:** $\quad \frac{d\hat{y}}{dz} = \sigma(z)(1-\sigma(z)) = \hat{y}(1-\hat{y})$

Step 3 — $\frac{d\ell}{d\hat{y}}$**:** $\quad \frac{d\ell}{d\hat{y}} = -\frac{y}{\hat{y}} + \frac{1-y}{1-\hat{y}}$

The Beautiful Cancellation

Putting it all together:

$$\frac{d\ell}{dw_i} = \underbrace{\left[\frac{-y}{\hat{y}} + \frac{1-y}{1-\hat{y}}\right]}_{\frac{d\ell}{d\hat{y}}} \cdot \underbrace{\hat{y}(1-\hat{y})}_{\frac{d\hat{y}}{dz}} \cdot \underbrace{x_i}_{\frac{dz}{dw_i}}$$

$$= \left[\frac{-y(1-\hat{y}) + (1-y)\hat{y}}{\hat{y}(1-\hat{y})}\right] \cdot \hat{y}(1-\hat{y}) \cdot x_i$$

$$= \left[-y + y\hat{y} + \hat{y} - y\hat{y}\right] \cdot x_i = (\hat{y} - y)\, x_i$$

$$\boxed{\frac{d\ell}{dw_i} = (\hat{y} - y)\, x_i}$$

This is **identical** in form to the linear regression gradient — the sigmoid derivative cancels out beautifully!

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Gradient Descent Update Rule

Gradient:

$$\frac{d\ell}{dw_i} = (\hat{y} - y)\, x_i \qquad \frac{d\ell}{dw_0} = (\hat{y} - y)$$

Update rule (gradient descent):

$$w_i \leftarrow w_i - \eta\,(\hat{y} - y)\, x_i \qquad w_0 \leftarrow w_0 - \eta\,(\hat{y} - y)$$

- If $\hat{y} > y$ (over-predicted) → gradient is positive → decrease weights
- If $\hat{y} < y$ (under-predicted) → gradient is negative → increase weights
- If $\hat{y} = y$ (perfect) → gradient is zero → no update

Training Algorithm

Step	Action
1	Initialize $w_1, \ldots, w_n, w_0$ randomly (or to zero)
2	Pick a random data point $(x_1^{(i)}, \ldots, x_n^{(i)}, y^{(i)})$
3	Compute $z = \mathbf{w}^T \mathbf{x}^{(i)}$
4	Compute prediction $y^{(i)} = \sigma(z)$
5	Update: $w_j \leftarrow w_j - \eta(y^{(i)} - y^{(i)})\, x_j^{(i)}$ for all j
6	Repeat from Step 2 until convergence

Stopping rule: when the loss change per iteration falls below a threshold ϵ.

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Extending to Multiple Classes

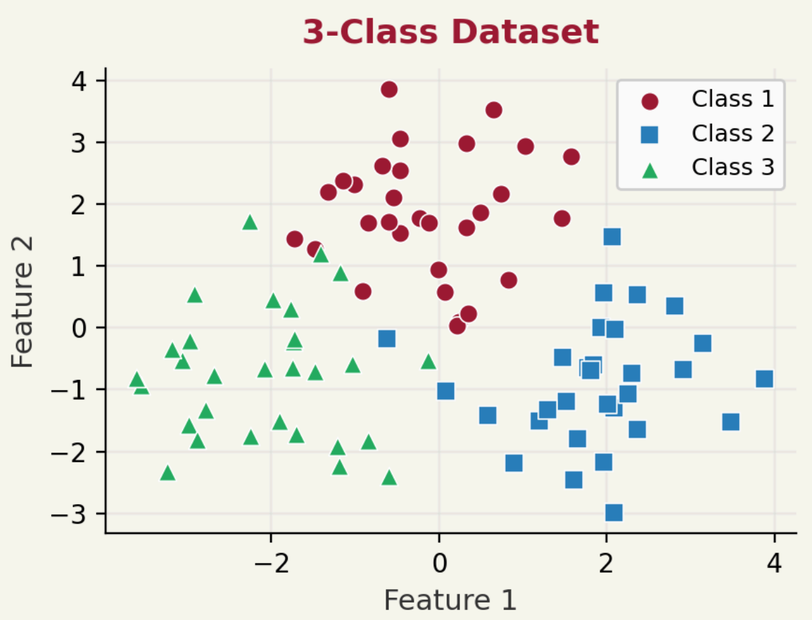
Logistic regression is inherently **binary** (two classes). For K classes, we use the **One-vs-All (OvA)** strategy.

Procedure:

1. Train K **separate binary classifiers**: one per class
2. Classifier k predicts: *"Is this example Class k or not?"*
3. Each outputs a probability $\hat{y_k} = \sigma(z_k)$
4. **Predict the class with the highest probability:**

$$\hat{y} = \underset{k \in \{1,\ldots,K\}}{\arg\max}\ \hat{y_k}$$

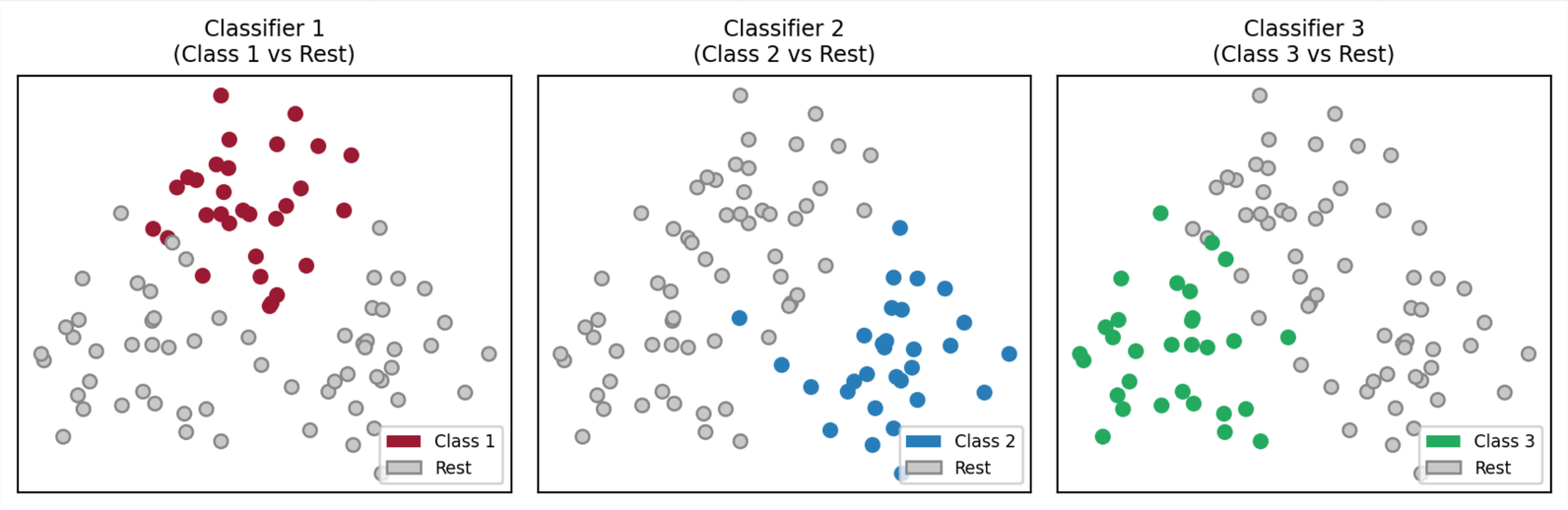
For $K = 3$ classes: train 3 classifiers and pick the most confident one.



One-vs-All: Diagram

For K classes: $\hat{y} = \underset{k \in \{1,\ldots,K\}}{\arg\max} \; f_k(x)$

One-vs-All: Visualization



Each classifier learns to separate **one class** from the rest. Final answer: class with **highest** $\hat{y_k}$.

Thank You!